Environmental efficiency among corn ethanol plants

Juan P. Sesmero a,*, Richard K. Perrin b, Lilyan E. Fulginiti b

a Department of Agricultural Economics, Purdue University, 403 W. State Street, KRAN 591A, West Lafayette, IN 47907-2056, USA
b Department of Agricultural Economics, University of Nebraska, Lincoln, NE 68583, USA

Abstract

Economic viability of the US corn ethanol industry depends on prices, technical and economic efficiency of plants and the extent of policy support. Public policy support is tied to the environmental efficiency of plants measured as their impact on emissions of greenhouse gases. This study evaluates the environmental efficiency of seven recently constructed ethanol plants in the North Central region of the US, using nonparametric data envelopment analysis (DEA). The minimum feasible level of GHG emissions per unit of ethanol is calculated for each plant and this level is decomposed into its technical and allocative sources. Results show that, on average, plants in our sample may be able to reduce GHG emissions by a maximum of 6% or by 2.94 Gg per quarter. Input and output allocations that maximize returns over operating costs (ROOC) are also found based on observed prices. The environmentally efficient allocation, the ROOC-maximizing allocation, and the observed allocation for each plant are combined to calculate economic (shadow) cost of reducing greenhouse gas emissions. These shadow costs gauge the extent to which there is a trade off or a complementarity between environmental and economic targets. Results reveal that, at current activity levels, plants may have room for simultaneous improvement of environmental efficiency and economic profitability.

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1. Introduction

The US corn ethanol industry has benefited from government support due to its potential to achieve a rather wide set of goals: mitigating emissions of greenhouse gases (GHG), achieving energy security (diversifying energy sources), improving farm incomes and fostering rural development among others. Continuation of policy support, however, is being debated due to doubts about the direct and indirect GHG effects of the industry. Moreover, the capacity of the industry to reduce GHG emissions per unit of output produced may also determine the opportunities opened to it in future carbon markets and in the National Renewable Fuel Standard program. This study provides information relevant to these issues by measuring the environmental performance of the industry in terms of GHG emissions per m³ produced and the economic cost (shadow price) of GHG reductions.

The input requirements and yield of byproducts are critical to determine the environmental performance have been addressed by some previous studies. Using engineering estimates [1], and [2] measured considerable improvement in plant efficiency between 2000 and 2006. Input requirements and cost data were reported by Ref. [3] based on a USDA sponsored survey of plants for the year 2002. Results were also reported by Refs. [4] and [5] based on spreadsheet models of the industry (GREET and BEACCON, respectively). Other studies reported average performances of plants although they did not clearly indicate the sources of their estimates [6]. Finally Ref. [7] reported results on input requirements, operating costs, and operating revenues based on a survey of seven

* Corresponding author. Tel.: +1 (765) 494 7545.
E-mail addresses: jsesmero@purdue.edu, juampase@hotmail.com (J.P. Sesmero).
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dry grind plants in the Midwest during 2006 and 2007. That study does not address the carbon footprint of the plants, but provides basic survey data for the present study.

The selection criteria for plants to be contacted for the sample in Ref. [7] were as follows. The plant must have started production (or been updated) after mid-2005 with a capacity of about 190 dam$^3$ per year or more, so as to represent recent technology. Plants must provide a minimum of three quarters of data, starting at least one month after the plant started operations (to work out initial inefficiencies). Finally, to facilitate companion studies of the impact of ethanol plants on rural communities, the plants had to be located in or near small towns of no more than 10,000 people. Eighteen plants that met these criteria were contacted in the process of obtaining one participating plant from each of seven states in the North Central region of the US (concerns about confidentiality and effort limited willingness to participate). The selection criteria nonetheless suggest that data from these plants would provide useful and representative information for evaluating the potential for GHG reductions within the current technology of the ethanol industry.

2. Materials and method

2.1. Data

The environmental performance of a plant is evaluated on the basis of emissions of greenhouse gases associated with the ethanol produced, as calculated by life cycle analysis (LCA) based on the use of inputs. The data are obtained from 33 quarterly reports of input and output quantities and prices from a sample of seven Midwest ethanol plants. Following the nonparametric efficiency literature, we refer to each observation as a decision making unit (DMU). Plants reported quarterly quantities and prices for 3 outputs: ethanol, dry distillers grains with solubles (DDGS, with 10% mass fraction of water or wB), and modified wet distillers grains with solubles (MWDGS wB = 55%). Prices and quantities of three inputs were reported (corn, natural gas, and electricity). Only total expenditures were reported for labor, denaturant, chemicals, and “other processing costs”. We therefore calculated implicit quantities for these inputs, dividing total expenditures by their corresponding price indexes. The labor and management price index associated to the Basic Chemical Manufacturing Industries was obtained from Ref. [8]. Implicit quantities of denaturant, chemicals and other processing inputs were similarly calculated based on the Producer Input Price Index for “All other basic inorganic chemicals” also obtained from Ref. [8]. No information about capital costs was collected as a part of the survey, so the economic analyses here are limited to evaluating returns over operating costs (ROOC).

2.2. Ethanol plants: characteristics

The surveyed plants produced an average rate equivalent to 201,000 m$^3$ of ethanol per year, with a range from 161,000 m$^3$ per year to 333,000 m$^3$ per year. The period surveyed included from the third quarter of 2006 until the fourth quarter of 2007 (six consecutive quarters). On average 54% of byproduct was sold as DDGS, but this ranged from one plant that sold absolutely no byproduct as DDGS to another plant that sold nearly all byproduct (97%) as DDGS. Descriptive statistics for inputs and outputs are shown in Table 1. Further information about the characteristics of these plants can be found in Ref. [7].

2.3. Environmental performance of ethanol plants

2.3.1. Emissions measurement

GHG emissions related to ethanol production are not directly measured, but instead are calculated from materials used over the “life cycle”. A number of computer packages has been developed to facilitate these calculations [4,9]. We used the Biofuels Energy Systems Simulator (BESS), developed at the University of Nebraska, Lincoln [10], which includes all GHG emissions from the burning of fossil fuels for crop production, grain transportation, the biorefinery, and coproduct transport. All upstream GHG emissions associated with the production of fossil fuels, fertilizer and electricity and for irrigation are included. For this analysis we used scenario 2 in BESS, “US Midwest average UNL”.

The BESS calculations of GHG emissions associated with a dry mill plant are equivalent to the following linear relationship:

$$\text{GHG}_e = 280x_c + 2.27x_{\text{nat}} + 740x_{\text{elec}} + 83.5u_{\text{eth}} - 495u_{\text{DDGS}} - 482u_{\text{MWDGS}}$$

where GHG$_e$ represents g of life cycle CO$_2$ equivalent greenhouse gases, $x_c$ is kg of corn used by the plant, $u_{\text{DDGS}}$ and $u_{\text{MWDGS}}$ are kg of byproduct sold as dried and modified wet respectively by the plant, $x_{\text{nat}}$ is the total amount of natural gas used by the plant measured in dm$^3$, $x_{\text{elec}}$ is total amount of GWh of electricity used by the plant, and $u_{\text{eth}}$ is the plant’s ethanol production in dm$^3$.

Eq. (1) states that 1 kg of corn used in a biorefinery is associated with about 280 g of GHG emitted during the production of that kg. DDGS and MWDGS have a positive and a negative component. The former is due to additional energy used in reducing moisture. In particular MWDGS require the use of electricity to centrifuge the wet byproduct and DDGS require the use of natural gas for heating and drying the wet byproduct after the centrifuge. The negative part is composed of “credits” attributed to byproducts (i.e. reductions in GHG) due to the replacement of corn that would have been fed to livestock had the byproduct not been sold. The “US Midwest average UNL” scenario in BESS assumes a corn yield of 9.57 Mg ha$^{-1}$. However data from USDA–NASS indicates that

<table>
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<th>Descriptive statistics: inputs and outputs.</th>
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average yield in this region was 8.65 Mg ha\(^{-1}\) during the period under analysis here. BESS allows adjustment of the yield assumption and the corn and byproduct coefficients in Eq. (1) are consistent with a corn yield of 8.65 Mg ha\(^{-1}\). The coefficient for ethanol production represents the combination of emissions associated with depreciable capital (54) and freight for grain transportation (29.5), expressed on a per m\(^3\) basis.

Eq. (1) can be expressed in vector notation. We partition inputs and outputs into a column vector of pollution-increasing variables \(\mathbf{a}^\prime = (x_1, x_{\text{DG}}, x_{\text{elect}}, u_{\text{Eth}}^\prime)\) and a column vector of pollution-reducing byproducts \(\mathbf{b}^\prime = (u_{\text{DDGS}}, u_{\text{PDGS}}^\prime)\). The level of greenhouse gas emissions associated with a particular plant \(j\) can now be expressed as

\[
\text{GHG} = \mathbf{a}^\prime \mathbf{x} + \beta \mathbf{b}^\prime
\]

where \(\mathbf{a} = (280, 2.27, 740, 83.5)\) is the \(1 \times 4\) vector of coefficients associated with pollution-increasing categories \(a^\prime\), and \(\beta = (-495, -482)\) is the \(1 \times 2\) row vector of coefficients associated with pollution-reducing byproducts \(\mathbf{b}^\prime\).

2.3.2. Characterization of potential ethanol technology from individual plant data

Plants are constrained by a technology transforming a vector of \(N\) inputs \(x = (x_1, x_2, \ldots, x_N)\in \mathbb{R}^N\) into a vector of \(M\) outputs \(u = (u_1, u_2, \ldots, u_M)\in \mathbb{R}^M\). Observed combinations of inputs used and outputs produced \((x^\prime, u^\prime)\) are taken to be points from the feasible ethanol technology. In this study we use data envelopment analysis (DEA) to infer the boundaries of the feasible technology set from the observed points, following Ref. [12].

Observations from the technology consist of a sample of \(33\) DMUs producing \(3\) outputs and using \(7\) inputs. We represent the production technology by a graph denoting the collection of all feasible input and output vectors

\[\text{GR} = \{ (x, u) \in \mathbb{R}^{7 \times 1} : x \leq N \} \]

where \(l(u)\) is the input correspondence which is defined as the collection of all input vectors \(x\in \mathbb{R}^N\) that yield at least output vector \(u\in \mathbb{R}^M\). The frontier of the graph \(GR\) and observed levels of inputs and outputs will serve as references for environmental efficiency assessment.

2.3.3. Environmental efficiency measurement

We use the calculated levels of GHG emissions corresponding to different DMUs (Eq. (1)) to calculate their environmental efficiency [11]. DMU \(j\) is deemed more environmentally efficient the lower are GHG emissions given the observed amount of ethanol production denoted by \(u_{\text{Eth}}^\prime\). Fixing ethanol production to its observed level, and assuming variable returns to scale and strong disposability of inputs and outputs, the graph describing technically feasible input and output combinations is denoted by

\[\text{GR} \left( V, S, u_{\text{Eth}}^\prime \right) = \left\{ (x, u) : u^\prime \leq z M_b, x^\prime \geq z N, z u_{\text{Eth}} = u_{\text{Eth}}^\prime, \sum_{j=1}^{33} v^j = 1, j = 1, \ldots, 33 \right\},\]

where \(z\) depicts a row vector of \(33\) intensity variables, \(M_b\) is the \(33 \times 2\) matrix of observed byproduct combinations, \(u^\prime\) is the \(1 \times 2\) vector of byproduct combinations observed for the \(j\)th DMU, \(N\) is the \(33 \times 7\) matrix of observed inputs, \(x^\prime\) is the \(1 \times 7\) vector of observed inputs for the \(j\)th DMU, \(u_{\text{Eth}}\) is the \(33 \times 1\) vector of observed outputs, and \(u_{\text{Eth}}^\prime\) is the ethanol production observed for DMU \(j\).

We define the set of all combinations of corn, gas, electricity and byproducts that result in lower emissions than those actually produced by the \(j\)th DMU as

\[
\text{GHG}^j \left( x_{\text{Eth}}, u_{\text{Eth}}^\prime, u_{\text{Eth}}^\prime \right) = \left\{ (x^\prime, u^\prime) : \alpha x^\prime + \beta u^\prime \leq \alpha x^\prime + \beta u_{\text{Eth}}^\prime \right\}
\]

subject to: \(u_{\text{Eth}}^\prime = u_{\text{Eth}}^\prime\)

where \(\alpha\) is a subset of the vector \(a\) previously defined which does not include the coefficient for ethanol, i.e. \(\alpha = (280, 2.27, 740)\) and the rest is as before.

From Eq. (4) we can derive an iso-pollution line in DDGS and corn space, i.e. combinations of DDGS and corn that result in the same level of emissions keeping everything else constant. Fig. 1 depicts the set defined in Eq. (4) graphically in the corn and DDGS space (i.e. keeping everything else in the GHG equation fixed). The set \(\text{GHG}^j\) consists of all those points above the iso-pollution line as indicated by the arrows with direction northwest.

In Fig. 1 the feasible technology set is represented by a graph displaying variable returns to scale and strong disposability of inputs and outputs as indicated by the arrows moving from the frontier \((u_{\text{DDGS}} = f(x))\) with direction southwest. As clearly seen in Fig. 1, the set \(\text{GHG}^j\) includes combinations outside the graph and hence not attainable by DMUs in the sample. The subset of observations in \(\text{GHG}^j\) that belong to the graph and are hence attainable by DMUs is depicted by the intersection of both sets delimited by the bold lines in Fig. 1.

![Fig. 1 – Decomposition of overall environmental efficiency.](image-url)
defined by equiproportional reductions in inputs and increases in byproducts. The value of the proportionate change necessary to encounter the boundary, $\text{ETE}_j^g$, is defined as the environmental technical efficiency of plant $j$

$$\text{ETE}_j^g(x_j^g, u_j^g, u_{\text{DDGS}}^g) = \min \left\{ \lambda : \text{GHG}_j \left( x_j^g, \lambda x_j^g, u_j^g, u_{\text{DDGS}}^g \right) \right\} \to 0$$

(6)

where $\lambda$ is a scalar defining the proportionate changes and the rest is as before. We calculated the value of $\text{ETE}_j^g(x_j^g, u_j^g, u_{\text{DDGS}}^g)$ using MATLAB as indicated in Appendix A.

Environmental technical efficiency defined in Eq. (6) is illustrated in Fig. 1 by the distance from $(x_j^g, u_{\text{DDGS}}^g)$ to point A which corresponds to the environmental technically efficient allocation in corn and DDGS space.

Note however that point A does not correspond to the minimum feasible GHG level since it does not coincide with the point of tangency between the iso-pollution and the graph (point B). The allocation that achieves the minimum level of GHG emissions subject to the graph is called the overall environmental efficient allocation.

Technically, we define this minimum feasible level of GHG emissions as

$$\text{GHG}_j^g = \min_{x_j^g, u_j^g} \left\{ \text{GHG}_j \left( x_j^g, u_j^g, u_{\text{DDGS}}^g \right) = a_1^j x_j^g + b_1^j \right\} \text{ s.t. } (x_j^g, u_j^g, u_{\text{DDGS}}^g) \in \text{GR} \left( V, S, u_{\text{DDGS}}^g \right)$$

(7)

where $\text{GHG}_j^g(x_j^g)$ denotes minimum emissions attainable by $j$ subject to observed ethanol production $u_{\text{DDGS}}^g$, $x_j^g$ is the vector of pollution-increasing inputs, $u_j^g$ is the vector of byproducts and the rest is as defined before. The empirical calculation of Eq. (7) is described in Appendix B.

Overall environmental efficiency, $E_j^g$, is measured by the hyperbolic distance between a given observation and the iso-pollution line corresponding to $\text{GHG}_j^g(x_j^g)$. The hyperbolic distance is computed through calculation of the reduction of observed inputs and equiproportional expansion of observed byproducts such that the iso-pollution corresponding to $\text{GHG}_j^g(x_j^g)$ is reached. This is illustrated by Fig. 1 where overall environmental efficiency is the distance between $(x_j^g, u_{\text{DDGS}}^g)$ and point C.

The hyperbolic movement from $(x_j^g, u_{\text{DDGS}}^g)$ to C results from the following technical relationship.

**PROPOSITION.** The measure of overall environmental efficiency, $E_j^g$, is related to minimum GHG in the following manner:

$$\text{GHG}_j^g = E_j^g a x_j^g + (E_j^g)^{-1} b p_j \quad j = 1, 2, ..., J$$

(8)

See Proof in Appendix C.

We can decompose $E_j^g$ into purely technical environmental efficiency $\text{ETE}_j^g$ (represented graphically by the distance between $(x_j^g, u_{\text{DDGS}}^g)$ and A) and environmental allocative inefficiency $\text{EAE}_j^g$ (represented graphically by the distance between A and O). Overall environmental efficiency can be expressed as

$$E_j^g = \text{EAE}_j^g \text{ETE}_j^g$$

(9)

Therefore, we can define allocative environmental inefficiency residually as $\text{EAE}_j^g = E_j^g / \text{ETE}_j^g$.

Environmental allocative inefficiency is illustrated in Fig. 1 by the distance between the iso-pollution corresponding to combination A and iso-pollution corresponding to point D. Based on the solution to the problem described in Eq. (7) we calculate overall environmental efficiency by solving the implicit Eq. (8) for each observation. These measures of environmental efficiency and their decomposition, Eq. (10), will be calculated for our sample of surveyed dry grind ethanol plants. The minimum feasible GHG for each DMU as defined by Eq. (7) is calculated fixing ethanol production at observed levels.

2.4. **ROOC and environmental targets: trade off or complementarity?**

From Eq. (2) there is a clear relationship between GHG and the combination of inputs and byproducts. But there is also a relationship between combinations of inputs and byproducts and the level of ROOC. Therefore, in general, a change in GHG levels through reallocation of inputs and byproducts would bring about a change in ROOC. For a given level of ethanol production, the shadow price of GHG mitigation is the change in ROOC per unit change in GHG levels. The change in ROOC denotes the plant’s maximum willingness to pay (WTP) for a permit to emit GHG. We define the shadow price of a unit of GHG as

$$SV_j^\text{GHG} = \frac{\text{WTP}}{\text{GHG}^j_0 - \text{GHG}^j_1} = \frac{\pi'^j - \pi'^0_j}{\text{GHG}^j_1 - \text{GHG}^j_0}$$

(11)

where WTP is willingness to pay for changing emissions from $\text{GHG}^j_0$ to $\text{GHG}^j_1$. $\text{GHG}^j_0$ denotes the original level of GHG and $\pi'^j_1$ the corresponding level of ROOC. $\text{GHG}^j_1$ is the “targeted” level of GHG and $\pi'^j_1$ denotes ROOC at this targeted level. GHG level will be targeted at the minimum GHG (i.e. $\text{GHG}^j_0 = \text{GHG}^j_1$), or alternatively at the level corresponding to maximum achievable ROOC by firm $j$, $\pi'^j_1$, which we designate as $\text{GHG}_j^j$. 

2.4.1. **Shadow cost from observed to ROOC-maximizing allocation**

We define the ROOC-maximizing combination of inputs and byproducts (subject to a given level of ethanol production to make it comparable with the GHG-minimizing combination) as the allocation that solves the following problem:

$$\pi'^j \left( r'^j, p'^j, r'^j_{\text{DDGS}}, \text{GR} \left( V, S, u_{\text{DDGS}}^g \right) \right) = \max_{x, u, p} \left\{ r'^j_{\text{DDGS}} + p' x \right\} \text{ s.t. } (x, u, p) \in \text{GR} \left( V, S, u_{\text{DDGS}}^g \right)$$

(12)

where $r'^j_{\text{DDGS}}$ is the observed price of ethanol obtained by observation $j$, $u_{\text{DDGS}}^g$ is the observed level of ethanol production by $j$, $u_b$ is the $2 \times 1$ column vector of variable outputs (DDGS and MWDGS), $r'$ represents the $1 \times 2$ vector of observed prices of variable outputs (byproducts) obtained by observation $j$ (since we did not have reported DDGS prices for three observations that did not sell DDGS we used average prices of DDGS obtained by other DMUs in the same quarter), $x$ is the $1 \times 7$ vector of variable inputs (corn, natural gas, electricity, labor, denaturant, chemicals, and “other processing costs”), and $p'$ represents the $1 \times 7$ vector of observed prices of variable.
inputs paid by \( j \). We will denote the allocation that solves Eq. (12) with ethanol fixed at the observed level by \( \{ \mathbf{x}^e, \mathbf{u}^e \} \). The level GHG \( \pi^j \) is calculated by inserting these values into (2).

We define the shadow value of GHG emissions associated with moving from the observed allocation to the ROOC-maximizing allocation as

\[
SV_{GHG}^j = \frac{\pi^j - \pi^i}{GHG_j^c - GHG_j^e}
\]  

(13)

An alternative shadow cost to Eq. (13) is that which is incurred by moving from the observed to the GHG-minimizing combination of inputs and byproducts.

2.4.2. Shadow cost from observed to GHG-minimizing allocation

The GHG-minimizing combination is computed by solving Eq. (7) with ethanol production fixed at observed levels and minimum GHG denoted by GHG\(^c\). ROOC associated with this allocation (calculated by multiplying the GHG-minimizing inputs and outputs times their respective prices) is designated as \( \pi^i/\pi^e \).

We define the shadow value of GHG related to a change from the observed to the GHG-minimizing point as

\[
SV_{GHG}^i = \frac{\pi^i - \pi^e}{GHG_j^c - GHG_j^e}
\]  

(14)

Finally we consider the shadow value of GHG related to a change from the GHG-minimizing to the ROOC-maximizing point.

2.4.3. Shadow cost from GHG-minimizing to ROOC-maximizing allocation

A change from the GHG-minimizing to the ROOC-maximizing allocation is illustrated in Fig. 2 in the corn and DDGS space. In Fig. 2 the GHG-minimizing combination is represented by point B (the iso-pollution line is denoted by GHG\(^c\)). If relative prices are those corresponding to the slope of \( \pi^i \) then ROOC maximization is achieved at point A and this requires a decrease in corn and DDGS with respect to the GHG-minimizing point. ROOC at A are denoted by \( \pi^i \) and ROOC at B are \( \pi^i < \pi^j \). Emissions at B are denoted by GHG\(^c\) and emissions at A are GHG\(^c\) > GHG\(^e\).

The shadow value associated with a change from the GHG-minimizing combination to the ROOC-maximizing one is defined by

\[
SV_{GHG}^i = \frac{\pi^i - \pi^i}{GHG_j^c - GHG_j^e}
\]  

(15)

3. Results and discussion

3.1. Environmental performance of ethanol plants

Fixing ethanol production at observed levels, measures of environmental efficiency and their decomposition are calculated for our sample of surveyed dry grind ethanol plants and reported in Table 2. Results reveal that DMUs are very efficient from a technical point of view and that most environmental inefficiency comes from allocative sources. Therefore DMUs seem to have room for GHG reductions mainly by changing input and output combinations subject to the graph. In particular, the average DMU may be able to reduce emissions by 6% which amounts to 2.944 Gg of CO\(_2\) equivalent GHGs per quarter.

The average DMU in our sample, at observed allocations, displays a GHG intensity of about 46 g CO\(_2\)e/MJ. At the GHG-minimizing allocation, the average DMU in our sample displays a GHG intensity as CO\(_2\)e mass of 43 g MJ\(^{-1}\) which is 6.5% lower than observed levels. This intensity is, for example, 55% lower than the target standard established by California by 2019 (86.3 g MJ\(^{-1}\)). It is of interest to know what reallocations of inputs and byproducts may actually achieve this improvement and we will go back to this point in detail later.

3.2. ROOC and environmental targets

Shadow costs associated with moving from observed to ROOC-maximizing allocations are reported in Table 3. Given the rather large variability across observations both the median and the average are reported as measures of central tendency. Table 3 displays some observations that are unusually high and others unusually low. These disproportionate deviations from the average are due to changes in inputs that affect ROOC but do not affect emissions, i.e. labor, denaturant, chemicals, and other processing costs. We classify as “outlier” any observation whose value exceeds the average by more than 3 times the standard deviation.

Since there seems to be a great deal of variability in shadow prices of GHG across DMUs we have plotted a histogram that shows the approximate distribution of these values in Fig. 3. The histogram does not take into account those observations deemed as outliers. We have superimposed to the histogram a normal density function that smoothes out the distribution.

An important conclusion we can extract from Fig. 3 and Table 3 is the fact that almost all DMUs reduce GHG emissions by moving from observed to maximum ROOC (negative shadow values). This suggests that, under our convexity assumptions, most DMUs (including the arithmetic average
and the mean of the normal density function) may be able to increase ROOC and reduce GHG simultaneously which would in turn imply that these DMUs face no trade off between economic and environmental goals at current combinations of inputs and byproducts.

The fact that DMUs can rearrange inputs and byproducts in such a way that they can both increase ROOC and reduce emissions prompts the following questions:

- What inputs are reduced or increased and which byproduct is reduced or increased in such a rearrangement?
- Why are plants not exploiting these reallocations that achieve greater ROOC?

The answer to the first question for the average plant is provided in Table 4. The average DMU would achieve greater ROOC and lower GHG simultaneously mainly by reducing the use of corn, natural gas, and electricity per unit of output produced and reducing the fraction of byproducts sold as DDGS (as opposed to MWDGS). A part of these reductions is achieved through elimination of inefficiencies that would take the DMUs to the technological frontier but for the most part they are achieved through rearrangements along the surface described by the boundary of the graph, Eq. (3).

Rearrangements displayed in Table 4 imply giving up DDGS to increase MWDGS and, consequently, reduce natural gas and other inputs. Our calculations show that, on average, DMUs would reduce DDGS production by 450 kg (dry matter basis) and increase MWDGS by 63 kg (dry matter basis). This implies that along the technology frontier reductions in inputs used per unit of output are associated with reductions in total production of byproducts per unit of output. These rearrangements are feasible in the sense that they achieve an allocation already achieved by some other DMU in the sample or a convex combination of allocations observed in the sample.

The answer to the second question is not as straightforward. As noted in the discussion of the first question our DMUs may be able to increase ROOC and reduce GHG mainly by reducing corn, natural gas, and electricity per unit of output produced and by reducing the fraction of byproducts sold as DDGS. This should not come as a surprise. Corn and natural

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<td>0.965</td>
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a This is calculated by taking the difference between calculated (based on observed inputs and outputs) and minimum GHG emissions.

b Reduction in GHG emissions from previous column as a percentage of calculated (based on observed inputs and outputs) emissions.
gas together amount to 80% of DMUs operating costs while byproducts only amount to 20% of revenue. Therefore DMUs find convenient to reduce corn and natural gas use per unit of output even if that means reducing byproducts produced per unit of output.

The apparent (in)ability to maximize ethanol yields and choose the right DDGS/MWDGS ratio when compared to other DMUs in the sample seems to drive the difference between observed production plans and ROOC-maximizing plans for many DMUs. However, a note of caution is in place here. There are many potential reasons for the failure of DMUs to attain the ROOC-maximizing allocation. First plants may not face market conditions that allow them to reallocate byproducts from dry to wet. A rather significant livestock production relatively near the plant has to be in place for DMUs to be able to sell a significant portion of their byproduct as well. These market constraints are not captured by our analysis. Second the graph is assumed to be convex in our calculations. Under the assumption of convexity any difference in performance is attributed to efficiency differences rather than to technological constraints. However there may be indivisibilities in the construction and later modifications.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Shadow value of GHG ($ Mg⁻¹) (observed to ROOC maximizing)</th>
<th>Shadow value of GHG ($ Mg⁻¹) (observed to GHG minimizing)</th>
<th>Shadow value of GHG ($ Mg⁻¹) (GHG minimizing to ROOC maximizing)</th>
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<tr>
<td>Median</td>
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Fig. 3 – Histogram of shadow values (observed to ROOC maximizing).
Shadow costs associated with moving from observed to GHG-minimizing allocations, Eq. (14), for each DMU, average, and median are reported in Table 3. Twelve DMUs lose ROOC while reducing GHGs, thus facing positive shadow values of GHGs, meaning a cost. Seventeen DMUs increase ROOC while reallocating to the minimum GHG level. The fact that the average willingness to pay for a change in allocation is positive while average change in GHG is negative, results in negative average shadow values. Table 3 indicates that the average DMU may be able to increase ROOC while reducing GHG which again seems to suggest unexploited opportunities to improve both fronts. In particular the average DMU may be able to increase ROOC by about $41 per Mg of GHG reduced. The seventeen firms with negative shadow prices would presumably be willing to sell permits at any small price, since there is no ROOC lost from reducing their own GHGs.

A histogram of these shadow prices (including a normal density function) is displayed in Fig. 4. The histogram does not include those observations deemed as outliers. The presence of outliers is mainly due, as discussed above, to changes in inputs affecting ROOC but not GHG, i.e. labor, denaturant, chemicals, and other processing costs. Despite the variability across DMUs, the highest frequency of shadow values (i.e. most of the “mass” of the distribution) appears to be located around zero. This means that plants are approximately efficient in the sense that they are operating at levels for which the marginal value of GHG is around zero which is, in turn, the current GHG price that DMUs face.

According to Table 5 the average DMU achieves minimization of GHG through substantial reductions in the fraction of byproducts sold as DDGS which in turn allows it to significantly reduce natural gas. Finally reductions in corn used per unit of output (achievable along the technology frontier through reductions in DDGS) are also important in GHG minimization. Such reallocations not only achieve reductions in GHG but also increase ROOC (negative shadow value).

Results in Table 5 are consistent with prior expectations. While corn production is responsible for about half of life cycle GHG emissions, natural gas and electricity used by the processing facility are responsible for 35–40% of life cycle emissions. On the other hand byproducts produced reduce life cycle GHG emissions by about 25%. To minimize GHGs, firms must achieve technical efficiency by reducing corn per unit of ethanol (even at the expense of lower byproduct production per unit of output) and reduce the fraction of byproduct sold as dry, which allows them to reduce natural gas use significantly.

Shadow costs associated with moving from GHG-minimizing to ROOC-maximizing allocations, Eq. (15), for each DMU, average and median are reported in Table 3. All DMUs increase both ROOC and GHGs in moving from low GHG solution to high ROOC solution. The average DMU would forfeit $1966 in ROOC for each Mg of GHG reduced, a very high cost of regulation if that firm were required to reduce GHGs. If DMUs are forced to reduce GHG emissions below ROOC-maximizing levels, these shadow values indicate that they would prefer to purchase permits if the market value is in the vicinity of $20–$30 per Mg, rather than reduce 1 Mg of GHG emissions. The histogram (with superimposed normal density) corresponding to values in Table 3 is plotted in Fig. 5. This histogram does not include outliers. Despite the variability across DMUs, the highest frequency of shadow values (i.e. most of the “mass” of the distribution) appears to be located around a very high value.

The reallocation of inputs and byproducts that would take the average DMU from the GHG-minimizing to the ROOC-maximizing combination is displayed in Table 6. The average DMU achieves increases in ROOC mainly through substantial

### Table 4 – Reallocation from observed to ROOC-maximizing combination.

<table>
<thead>
<tr>
<th>Category measure</th>
<th>Corn</th>
<th>Natural gas</th>
<th>Electricity</th>
<th>Percentage of byproduct sold as DDGS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average change (%)</td>
<td>-5.88</td>
<td>-3.83</td>
<td>-0.41</td>
<td>-7.35</td>
</tr>
</tbody>
</table>

### Table 5 – Reallocation from observed to GHG-minimizing combination.

<table>
<thead>
<tr>
<th>Category measure</th>
<th>Corn</th>
<th>Natural gas</th>
<th>Electricity</th>
<th>Percentage of byproduct sold as DDGS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average change (%)</td>
<td>-2.82</td>
<td>-5.95</td>
<td>-0.98</td>
<td>-15</td>
</tr>
</tbody>
</table>

![Fig. 4 – Histogram of shadow values (observed to GHG minimizing).](image-url)
increases in the fraction of byproducts sold as DDGS which in turn entails increases in natural gas and reductions in MWDGS. Another very important component of ROOC increases is reductions of corn per unit of output produced. Our calculations reveal that, on average, DDGS production would increase by 1.36 Mg (dry matter basis) and MWDGS production would decrease by 1.81 Mg (dry matter basis). Therefore, on average, reductions in corn used per unit of output produced are achieved by giving up total byproducts produced per unit of output. However since corn amounts to about 70% of operating cost and byproduct to 20% of revenue DMUs find convenient to reduce the former even if that means also reducing the latter.

Results for the average DMU in Table 3 can be combined to recover the shape of the relationship between GHG and ROOC. Plotting the three averages in the GHG and ROOC space yields the graph in Fig. 6. We denote the observed combination of the average by \((GHG_j^*, \pi^*)\), the ROOC-maximizing combination by \((GHG^*, \pi_j^*)\), and the GHG-minimizing combination by \((GHG_j^*, \pi_j^*)\). There seems to be room for simultaneous improvement of environmental and economic performance, as previously indicated in discussions of results in Table 3. However, if the average firm was able to adjust inputs and byproducts from the GHG-minimizing to the ROOC-maximizing combination, it would face an intense trade off described just above.

4. Conclusions

The purpose of this study was to contribute to the ongoing debate regarding the merits and potential of the ethanol industry in the US by investigating the current environmental performance at the individual plant level, the potential for improvement in this performance and its effects on the industry’s overall emissions of greenhouse gases.

Several important conclusions can be drawn from this study. First, our results suggest that decision making units (DMUs) may have some room for improving environmental performance. However since plants are technically very efficient, most of this improvement has to come from changes in combinations of inputs and byproducts along the frontier (reduction in environmental allocative inefficiencies). By eliminating allocative inefficiencies the average DMU could apparently decrease emissions by 6%, which amounts to about 2.944 Gg of CO2 equivalent GHG.

Negative shadow values of GHG from observed to ROOC-maximizing combinations reveal that at current operating levels DMUs may be able to increase ROOC and reduce GHG simultaneously by reaching the “best practice” in the sample. Plants may not be switching to the ROOC-maximizing combination because of capital costs involved in that reallocation. If such costs exist they are not being accounted for here. However these costs may be outweighed by revenue opportunities created through carbon reducing policies, e.g. renewable fuel standards, carbon markets, tax credits for carbon reducing capital investments, etc.

Additionally once DMUs achieve the ROOC-maximizing allocation, our results suggest that they may face significant ROOC losses if they are forced to reduce GHG any further. The average DMU would achieve this reduction mainly by increasing the fraction of byproducts sold as MWDGS because this allows it to reduce natural gas usage. In this case the average DMU in this sample would be willing to pay up to $1966 for a permit to emit 1 Mg of GHG, rather than suffer the
ROOC reduction revealed by the shadow price of reducing carbon from ROOC-maximizing to GHG-minimizing levels.

The measurement of corn ethanol plants’ environmental performance, their potential for improvement, and ROOC/emissions trade offs conducted in this study should inform the debate on whether there is a place for corn ethanol as a “clean” substitute for gasoline. In particular our results suggest that ethanol plants in our sample can produce energy with considerably lower (52% lower) GHG intensity than gasoline. Moreover these plants have some room for reducing this footprint even more by reallocating inputs and byproducts. Such reallocations would achieve a 6% reduction in GHG rendering energy with a GHG intensity 55% lower than gasoline. A substantial fraction of these reductions may be achieved at a moderate or zero economic cost as suggested by a negative shadow price of $41 per Mg. Further reductions, however, can only be achieved at high economic costs.

Acknowledgments

Study supported by the Agricultural Research Service, University of Nebraska, and USDA regional project NC506.

Appendix A

The measure in (6) can be mathematically implemented through the following nonlinear programming problem:

\[
\begin{align*}
\min_{\lambda} & \quad \frac{1}{2} u_{gj}^T M_{g} z_j, \quad u_{gj}^T = 2 M_{\text{Eth}}, \quad \lambda^T x_j \geq N_j, \quad \sum_j z_j = 1
\end{align*}
\]  

(A.1)

where \( u_{gj} \) is the vector of dried and wet byproducts, \( M_{g} \) is the \( 2 \times J \) matrix of observed levels of byproducts, \( z_j \) is the \( J \times 1 \) vector of intensity variables used to weight observations and construct the piecewise linear boundary of the graph, \( x_j \) is the column vector composed by observed values of all inputs used by observation \( j \), \( N_j \) is the \( 7 \times J \) matrix of observed values of inputs for all observations, and \( u_{gj}^T \) is the observed level of ethanol production of the \( j \)th DMU.

After multiplying the constraints times \( \lambda \) it is easily seen that this is equivalent to the following problem:

\[
\begin{align*}
\min_{\lambda} & \quad 
\lambda^T z_j, \quad \lambda^T \geq N_j, \quad \sum z_j = \lambda, \\
\text{s.t.} & \quad u_{gj}^T M_{\text{Eth}} z_j = \lambda, \\
& \quad \lambda^T z_j = \lambda z, \quad z = \lambda z
\end{align*}
\]  

(A.2)

Following Ref. [12] problem (A.1) is reformulated into problem (A.2) because the only nonlinear constraint is an equality constraint (i.e. \( \lambda = \lambda^2 \) and is, hence, easier to program. In particular, these sub vector hyperbolic measures of technical efficiency are calculated through a nonlinear program implemented with the FMINCON procedure in MATLAB.

Appendix B

The following program describes the problem:

\[
\begin{align*}
\min_{u_{\text{DDGS}}, u_{\text{MWDGS}}} & \quad \text{GHG}_{\text{Mg}} = 280 x_0 + 2.27 x_{NG} + 740 x_{\text{elect}} + 83.5 x_{\text{eth}} - 495 u_{\text{DDGS}} - 482 u_{\text{MWDGS}} \\
\text{s.t.} & \quad u_{\text{DDGS}} \leq M_{\text{DDGS}} Z, \quad u_{\text{MWDGS}} \leq M_{\text{MWDGS}} Z, \quad u_{\text{Eth}} = M_{\text{Eth}}, \quad x > N, \quad \sum_j z_j = 1
\end{align*}
\]  

(B.1)

Appendix C

Proof. Let us denote the vector of coefficients of Eq. (1) by \( (\alpha, \beta) \), where \( \alpha \) is the vector of coefficients for corn, natural gas, and electricity, and \( \beta \) is the vector of coefficients for both byproducts. In addition, let us define an arbitrary output and input vector by \( (x_p, u_b) \) where \( x_p = (x_{NG}, x_{\text{elect}}, x_{\text{eth}}) \) and \( u_b = (u_{\text{DDGS}}, u_{\text{MWDGS}}) \) and denote the \( j \)th DMU’s observed output and input vector by \( (x_{pj}, u_{bj}) \). Let \( (x_p, u_b) \in \text{GHG}_{GR}^{ij} \left( E_{p}^{j} \right) \left( E_{b}^{j} \right) \in \text{GR} \) then \( (x_p, u_b) \in \text{GR} \) and since \( E_{p}^{j} \) is a minimum

\[
\begin{align*}
(\alpha x_p + \beta u_b) = E_{p}^{j} \left[ 280 x_0 + 2.27 x_{NG} + (740) x_{\text{elect}} \\
- 495 u_{\text{DDGS}} - 482 u_{\text{MWDGS}} \right]
\end{align*}
\]

Let us denote observations \( j \)'s minimum feasible GHG level by \( \text{GHG}_{GR}^{j} \). There are three cases to consider:

1. Assume \( (\alpha x_p + \beta u_b) < \text{GHG}_{GR}^{j} \), then \( (x_p, u_b) \notin \text{GR} \)
2. Assume \( (\alpha x_p + \beta u_b) > \text{GHG}_{GR}^{j} \), then \( (v, w) \in \text{GR} \) and since the hyperplanes defining the two sets are parallel, \( E_{b}^{j} \) cannot be a minimum.

Cases 1 and 2 leave the following case:

3. \( (\alpha x_p + \beta u_b) = \text{GHG}_{GR}^{j} \). Therefore \( E_{p}^{j} \alpha x_p + E_{b}^{j} \beta u_b = \text{GHG}_{GR}^{j} \).
REFERENCEs


